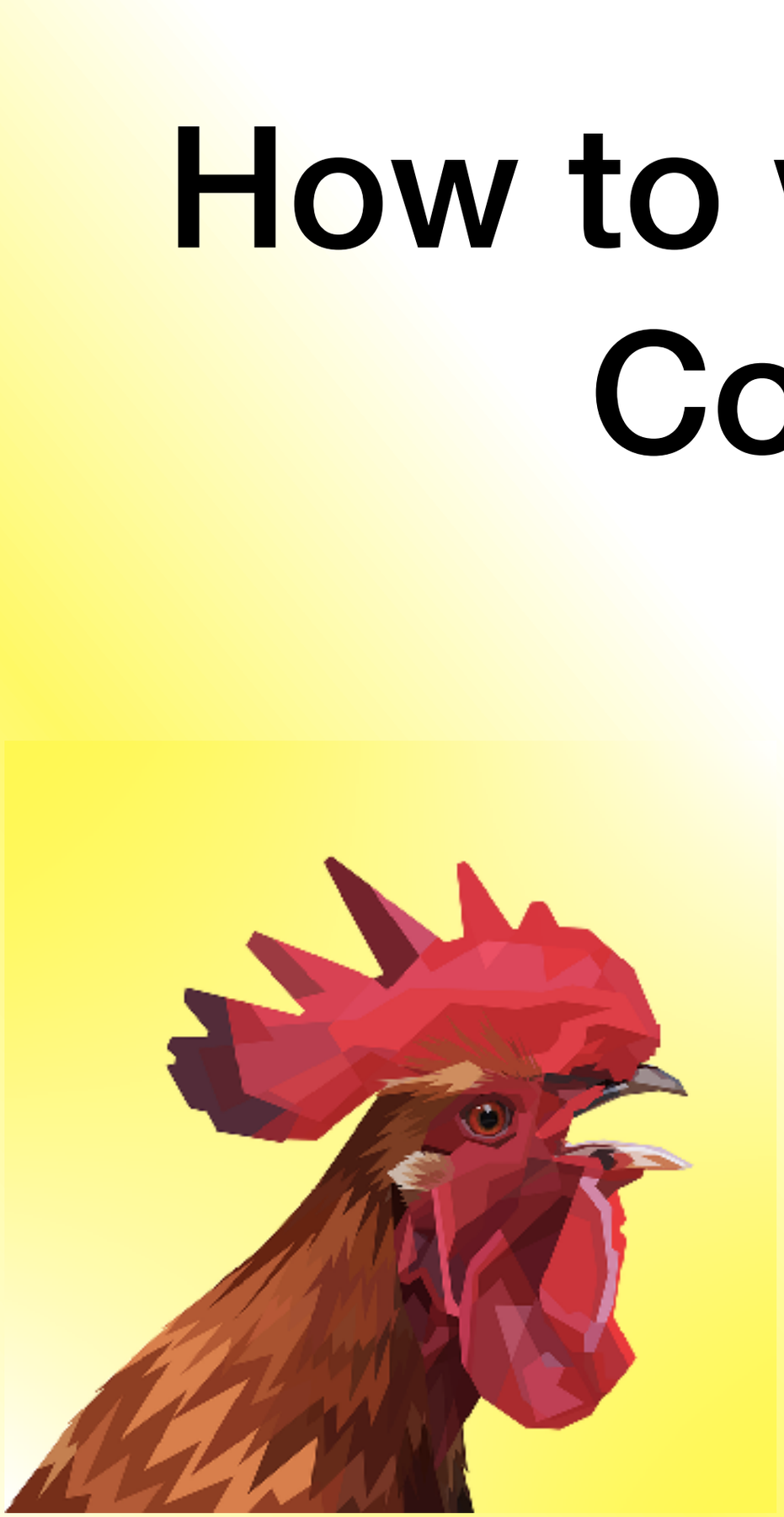
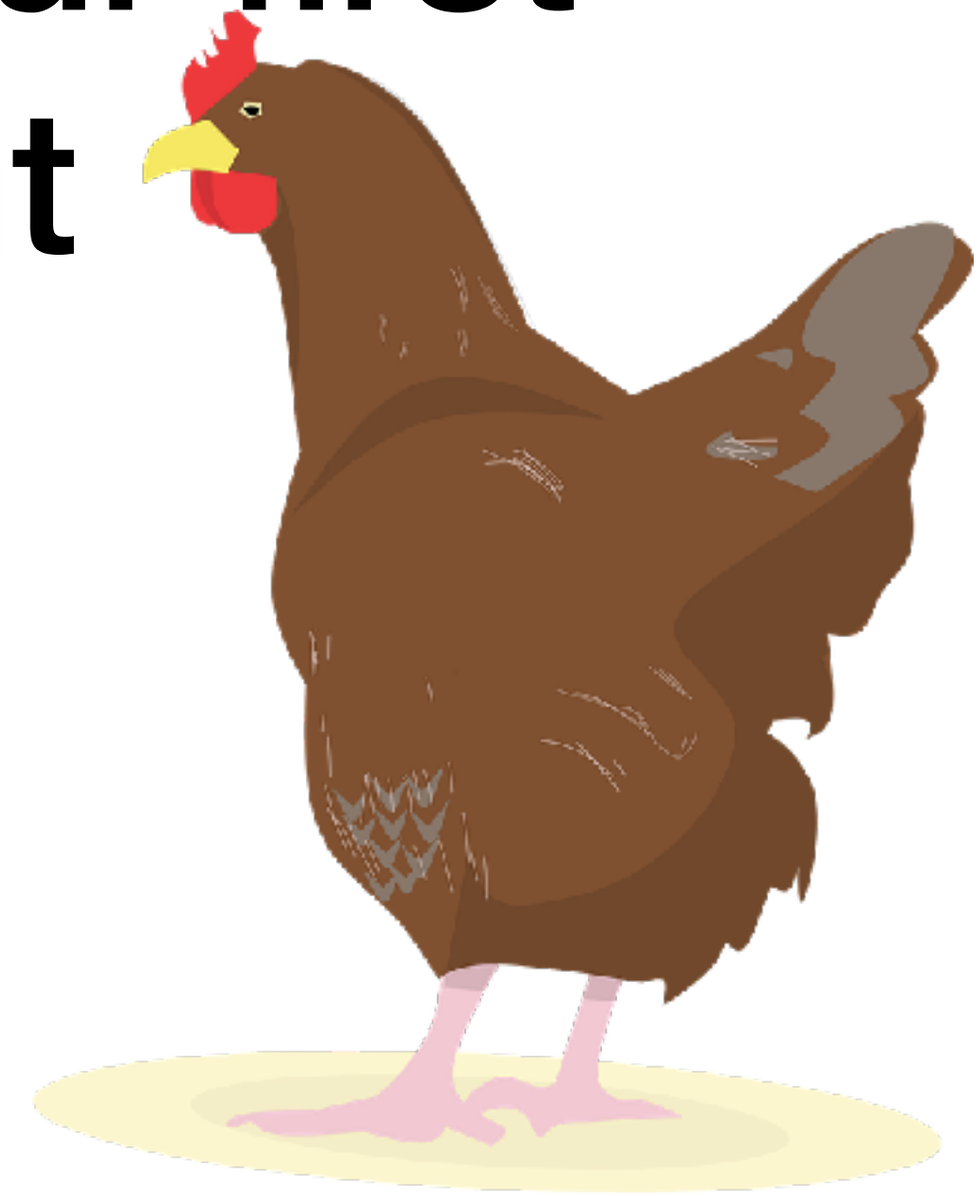
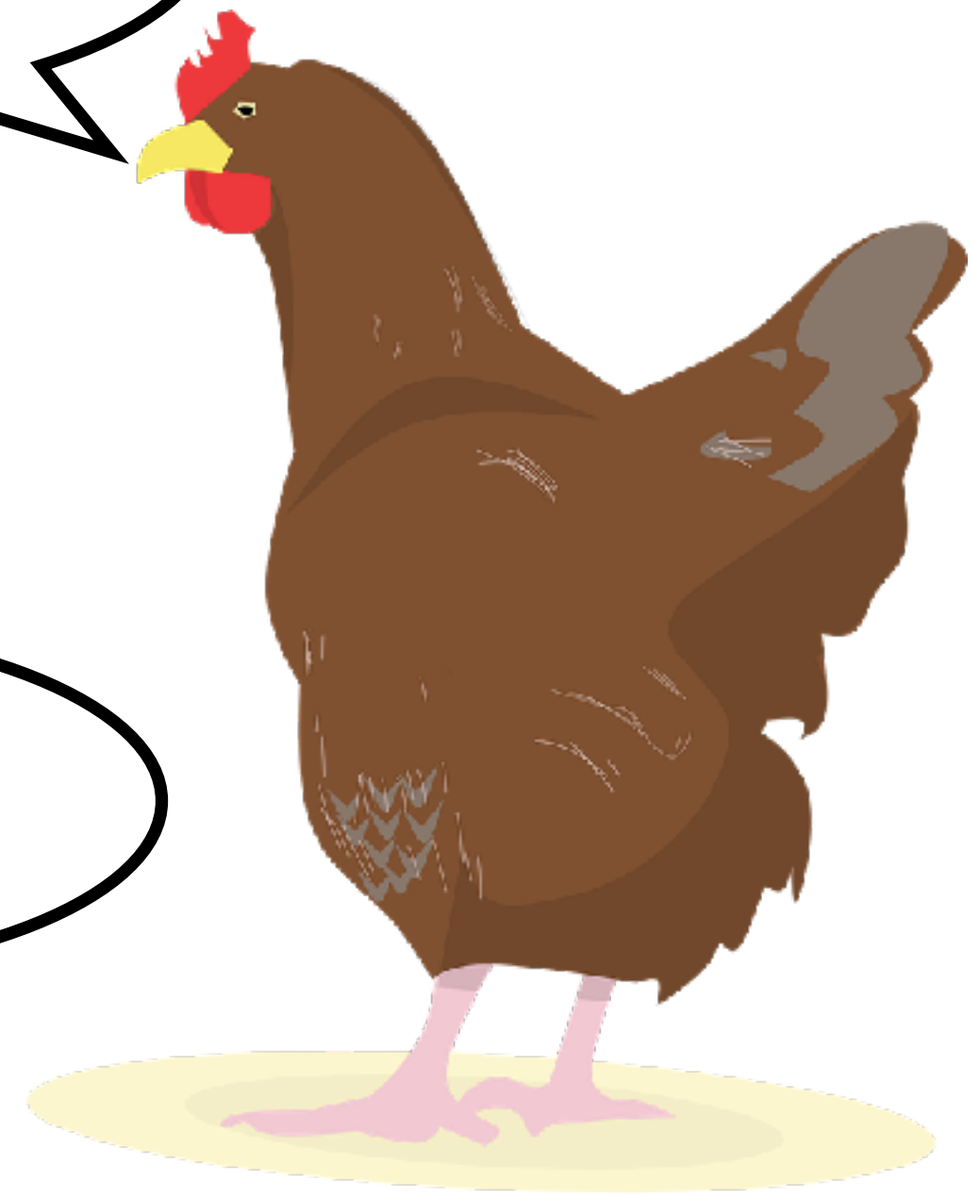
How to win your first Coq Fight





Induction!



QED!

Assertions

$$A \vee B, \quad A \rightarrow C, \quad B \rightarrow D \quad \vdash \quad C \vee D$$

what to prove

$$A \vee B,\quad A \to C,\quad B \to D \quad \vdash \quad C \vee D$$

what we can assume

$$A \vee B, \quad A \to C, \quad B \to D \quad \vdash \quad C \vee D$$

Inference rule

$$\frac{\Gamma,\ x \vdash z \qquad\qquad \Gamma,\ y \vdash z}{\Gamma,\ x \vee y \vdash z}\ \text{(destruct V)}$$

for all Γ, x, y, z

$$\frac{\Gamma,\ x \vdash z \qquad\qquad \Gamma,\ y \vdash z}{\Gamma,\ x \vee y \vdash z}\ \text{(destruct V)}$$

$$\frac{\Gamma,\ x \vdash z \qquad\qquad \Gamma,\ y \vdash z}{\Gamma,\ x \vee y \vdash z}\ \text{(destruct V)}$$

the rule allows us to prove

by proving these instead

$$\frac{\Gamma,\ x \vdash z \qquad\qquad \Gamma,\ y \vdash z}{\Gamma,\ x \vee y \vdash z}\ \text{(destruct } \vee\text{)}$$

Instantiate the rule

$$\frac{\Gamma,\ x \vdash z \qquad\qquad \Gamma,\ y \vdash z}{\Gamma,\ x \vee y \vdash z}\ \text{(destruct } \vee\text{)}$$

Instantiate the rule

$$\frac{\Gamma,\ x \vdash z \qquad\qquad \Gamma,\ y \vdash z}{\Gamma,\ x \vee y \vdash z}\ \text{(destruct V)}$$

$$\frac{}{A \vee B,\ A \to C, B \to D \vdash C \vee D}$$

Instantiate the rule

$$\frac{\Gamma,\ x \vdash z \qquad\qquad \Gamma,\ y \vdash z}{\Gamma,\ x \vee y \vdash z}\ (\text{destruct } \vee)$$

$$\frac{A,\ A \to C, B \to D \vdash C \vee D \qquad\qquad B,\ A \to C, B \to D \vdash C \vee D}{A \vee B,\ A \to C, B \to D \vdash C \vee D}\ (\text{destruct } \vee)$$

$$\dfrac{\dfrac{}{A,\ A\to C, B\to D \vdash C \vee D} \qquad \dfrac{}{B,\ A\to C, B\to D \vdash C \vee D}}{A \vee B,\ A\to C, B\to D \vdash C \vee D}\ (\text{destruct } \vee)$$

We need more rules

$$\frac{\Gamma \vdash x}{\Gamma \vdash x \vee y}\ \text{(left } \vee\text{)}$$

$$\frac{\dfrac{}{A, A \to C, B \to D \vdash C \vee D} \qquad \dfrac{}{B, A \to C, B \to D \vdash C \vee D}}{A \vee B, A \to C, B \to D \vdash C \vee D}\ \text{(destruct } \vee\text{)}$$

We need more rules

$$\frac{\Gamma \vdash x}{\Gamma \vdash x \vee y}\ \text{(left } \vee\text{)}$$

$$\dfrac{\dfrac{A, A\to C, B\to D \vdash C}{A, A\to C, B\to D \vdash C \vee D}\ \text{(left } \vee\text{)} \qquad \dfrac{}{B, A\to C, B\to D \vdash C \vee D}}{A \vee B, A\to C, B\to D \vdash C \vee D}\ \text{(destruct } \vee\text{)}$$

We need more rules

$$\frac{\Gamma \vdash x \to y \qquad \Gamma \vdash x}{\Gamma \vdash y}\ \text{(apply)}$$

$$\dfrac{\dfrac{\dfrac{}{A,\ A \to C, B \to D \vdash C}}{A,\ A \to C, B \to D \vdash C \vee D}\ \text{(left } \vee\text{)} \qquad \dfrac{}{B,\ A \to C, B \to D \vdash C \vee D}}{A \vee B,\ A \to C, B \to D \vdash C \vee D}\ \text{(destruct } \vee\text{)}$$

We need more rules

$$\frac{\Gamma \vdash x \to y \qquad \Gamma \vdash x}{\Gamma \vdash y}\ \text{(apply)}$$

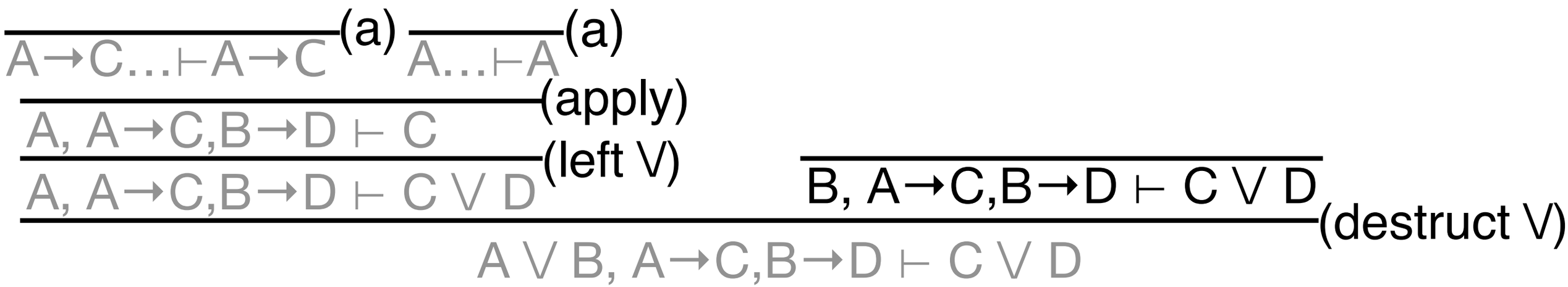
$$\dfrac{\dfrac{\dfrac{A \to C \ldots \vdash A \to C \qquad A \ldots \vdash A}{A,\ A \to C, B \to D \vdash C}\ \text{(apply)}}{A,\ A \to C, B \to D \vdash C \vee D}\ \text{(left } \vee\text{)} \qquad \dfrac{}{B,\ A \to C, B \to D \vdash C \vee D}}{A \vee B,\ A \to C, B \to D \vdash C \vee D}\ \text{(destruct } \vee\text{)}$$

We need more rules

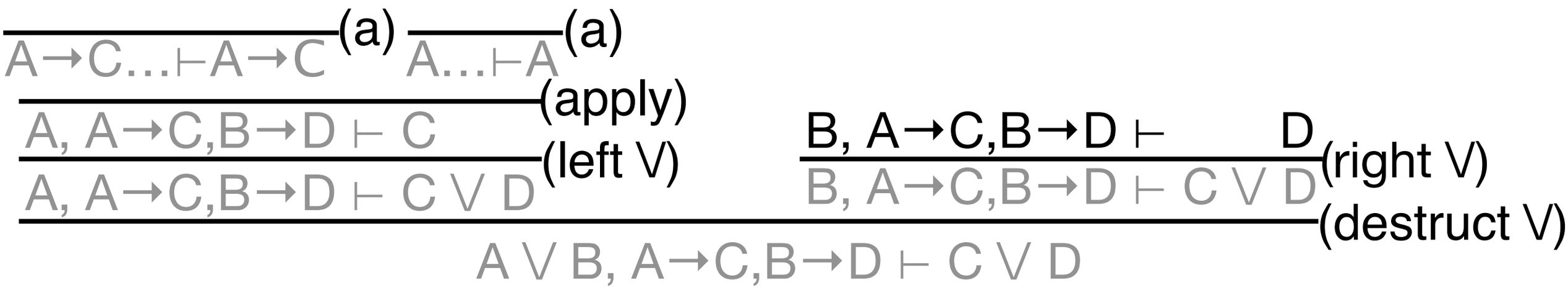
$$\frac{}{x, \Gamma \vdash x}\ \text{(assumption)}$$

$$\frac{\dfrac{}{A \to C, \ldots \vdash A \to C}\ \text{(a)} \quad \dfrac{}{A, \ldots \vdash A}\ \text{(a)}}{\dfrac{A, A \to C, B \to D \vdash C}{A, A \to C, B \to D \vdash C \vee D}\ \text{(left } \vee\text{)} \quad \dfrac{}{B, A \to C, B \to D \vdash C \vee D}\ \text{(left } \vee\text{)}}{A \vee B, A \to C, B \to D \vdash C \vee D}\ \text{(destruct } \vee\text{)}$$

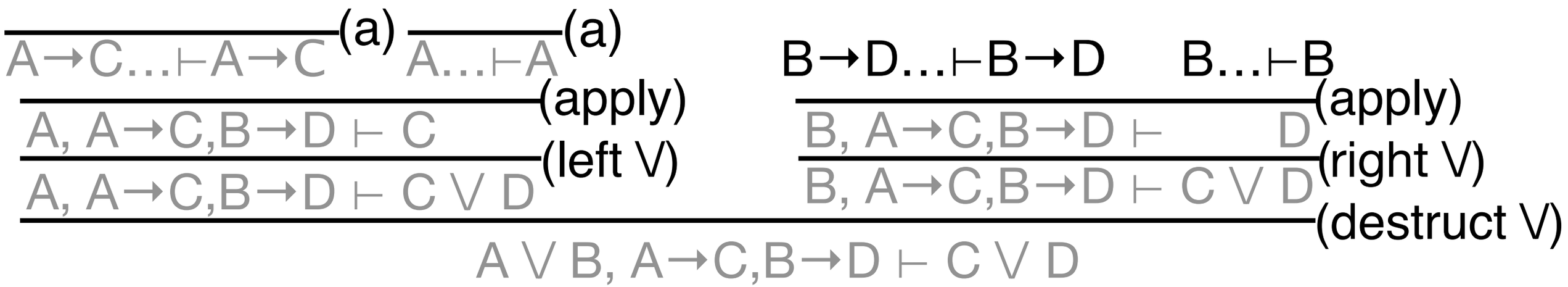
And so on


A→C…⊢A→C
(a)
A…⊢A
(a)
A, A→C,B→D ⊢ C
(apply)
A, A→C,B→D ⊢ C ∨ D
(left ∨)
B, A→C,B→D ⊢ C ∨ D
A ∨ B, A→C,B→D ⊢ C ∨ D
(destruct ∨)

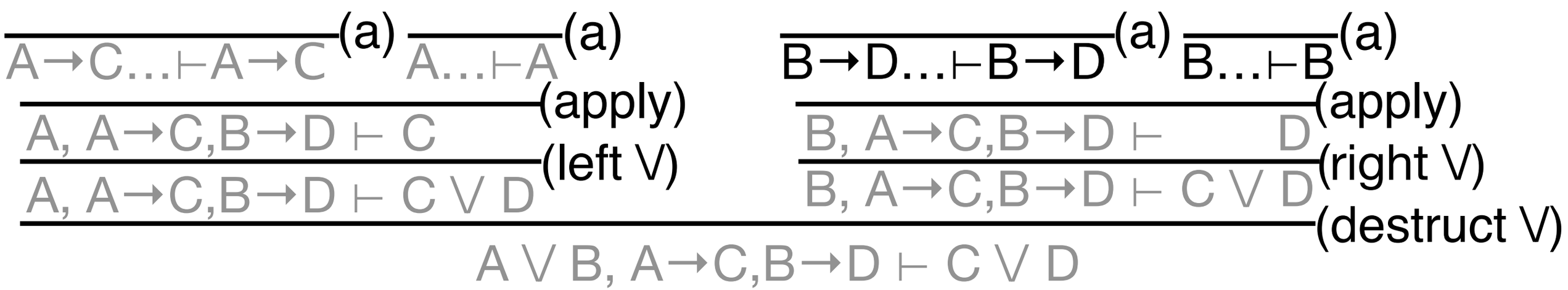
And so on


(a)
A→C…⊢A→C
(a)
A…⊢A
(apply)
A, A→C,B→D ⊢ C
(left ∨)
A, A→C,B→D ⊢ C ∨ D
B, A→C,B→D ⊢ D
(right ∨)
B, A→C,B→D ⊢ C ∨ D
(destruct ∨)
A ∨ B, A→C,B→D ⊢ C ∨ D

And so on


(a)
A→C…⊢A→C
(a)
A…⊢A
(apply)
A, A→C,B→D ⊢ C
(left ∨)
A, A→C,B→D ⊢ C ∨ D
B→D…⊢B→D
B…⊢B
(apply)
B, A→C,B→D ⊢ D
(right ∨)
B, A→C,B→D ⊢ C ∨ D
(destruct ∨)
A ∨ B, A→C,B→D ⊢ C ∨ D

And so on


A→C…⊢A→C (a)
A…⊢A (a)
B→D…⊢B→D (a)
B…⊢B (a)
A, A→C,B→D ⊢ C (apply)
B, A→C,B→D ⊢ D (apply)
A, A→C,B→D ⊢ C ∨ D (left ∨)
B, A→C,B→D ⊢ C ∨ D (right ∨)
A ∨ B, A→C,B→D ⊢ C ∨ D (destruct ∨)

In Coq

```
Inductive or (A B : Prop) : Prop :=
  | or_introl : A → or A B
  | or_intror : B → or A B
```

As rules

```
Inductive or (A B : Prop) : Prop :=
  | or_introl : A → or A B
  | or_intror : B → or A B
```

$$\frac{\Gamma,\ x \vdash z \qquad \Gamma,\ y \vdash z}{\Gamma,\ x \vee y \vdash z}\ \text{(destruct)}$$

$$\frac{\Gamma \vdash x}{\Gamma \vdash x \vee y}\ \text{(apply or\_introl)} \qquad \frac{\Gamma \vdash y}{\Gamma \vdash x \vee y}\ \text{(apply or\_intror)}$$

Let's prove it

Induction over lists

```
Inductive List (A : Type) : Type :=
  | Nil  :                    List A
  | Cons : A -> List A -> List A
```

$$\frac{\Gamma \vdash \mathrm{IH}(\mathrm{Nil}) \qquad \Gamma, a : A, as' : \mathrm{List}\ A, \mathrm{IH}(as') \vdash \mathrm{IH}(\mathrm{Cons}\ a\ as')}{\Gamma, as : \mathrm{List}\ A \vdash \mathrm{IH}(as)}$$

Where to learn more

- Software Foundations (Benjamin Pierce et al)

- https://softwarefoundations.cis.upenn.edu/